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Abstract

This paper considers the spectral determinant of quantum graph families with
chaotic classical limit. The secular coefficients of the spectral determinant
are found to follow distributions with zero mean and variance approaching a
constant in the limit of large network size for graphs without symmetries. This
constant is, in general, different from the random matrix result and depends on
the classical limit. A closed expression for this system-dependent constant is
given here explicitly in terms of the spectrum of an underlying Markov process.
Related results for graphs with time-reversal symmetry are given.

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1. Introduction

Quantum graphs and quantum networks have become a popular tool to model quantum
dynamics in mesoscopic systems in the diffusive regime (Shapiro 1982, Chalker and
Coddington 1988, see also Dittrich (1996) and Janssen (1998) for recent review papers),
as well as to study spectral statistics (Kottos and Smilansky 1997, 1999, Pakoński *et al*
2001, Ketzmerick *et al* 2000, Tanner 2001). In this paper, the autocorrelation function of
the spectral determinant

$$Z(\theta) = e^{-\frac{i}{2}(N\theta+\varphi)} \det(\mathbf{1} - e^{i\theta}\mathbf{U}) = e^{-\frac{i}{2}(N\theta+\varphi)} \sum_{n=0}^{N} a_n e^{in\theta} \tag{1}$$

is studied after averaging over a suitable ensemble of quantum graphs preserving the underlying
'classical' dynamics on the graph. Here, $\mathbf{U}$ is a unitary $N \times N$ matrix which describes the
quantum evolution on the graph and the a_n denote the secular coefficients of the characteristic
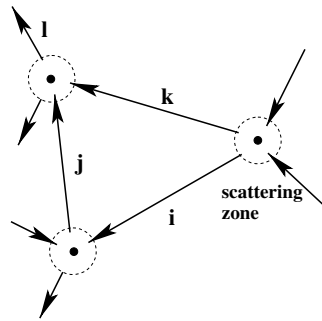
Figure 1. A section of a typical directed graph; waves travel along directed edges $i, j, k, l, \ldots$ undergoing scattering at the vertices.

polynomial. The phase factor in front of the determinant ensures that Z is a real function for real θ after setting $\exp(\mathrm{i}\varphi) = \det(-U)$ and using the basic property for the secular coefficients

$$a_n = \mathrm{e}^{\mathrm{i}\varphi} a_{N-n}^* \tag{2}$$

which follows from the unitarity of U. The autocorrelation function of (1) has been studied by Kuś *et al* (1993) and Haake *et al* (1996) after averaging over circular-orthogonal, -unitary and -symplectic ensembles of unitary matrices. Related results for ensembles of Hermitian matrices have been given by Kettemann *et al* (1997). What all these approaches have in common is that the average is taken over a set of matrices too large to link the ensemble in the semiclassical limit to a specific classical system. In this sense, the results are of a random matrix type. There is strong evidence, however, that the statistical properties of spectra related to a fixed chaotic classical system already follow the random matrix theory (RMT). This suggests that the size of the ensembles considered can be reduced drastically without losing universality in the spectral statistics.

Studies linking the autocorrelation function (1) for quantum systems to the properties of underlying classical dynamics have been presented by Kettemann *et al* (1997) and Smilansky (1997) for quantum maps, Keating (1996), Cheung (1997) and Snaith (2000) for general quantum systems and Kottos and Smilansky (1999) for quantum graphs. I will approach this problem here by defining unitary matrix ensembles corresponding to a specific class of quantum graphs. The autocorrelation function of the spectral determinant is studied for these ensembles. It will be shown that the variance of the secular coefficients a_n indeed follows the random matrix result for chaotic classical dynamics up to a multiplicative factor. This factor is system dependent and will be explicitly derived in section 4. This improves the formula given by Kettemann *et al* (1997) and Smilansky (1997) by carefully including repetitions of periodic orbits leading to non-negligible contributions. Related results have been reported by Keating (1996), Cheung (1997) and Snaith (2000) using periodic orbit approximations of the spectral determinant.

2. Quantum graphs and unitary stochastic ensembles

In the following I will use a definition of quantum graphs which is slightly more general than that introduced by Kottos and Smilansky (1997). Consider a graph consisting of N directed edges or bonds connecting an unspecified number of vertices. Free, one-dimensional wave propagation occurs along edges i of length L_i. Transitions from an edge i to an edge j is possible at a vertex if and only if i is an incoming edge and j an outgoing edge at the same vertex, see figure 1. The scattering process at a given vertex is described by a scattering

amplitude s_{ij}, where i and j label again incoming and outgoing edges, respectively. To keep the set of possible networks as general as possible, the local scattering amplitudes s_{ij} will not be specified any further other than to fulfill basic properties such as probability conservation. (One may think of the scattering event taking place at a given vertex to be due to an interaction with a complex short-range potential or a many-particle compound at that point.)

The eigenstates (or resonances) of such a network are given by the stationary states of the system. These are obtained by considering the N-dimensional wave-vector $\mathbf{\Psi}_{\text{out}}$ where the ith component of $\mathbf{\Psi}_{\text{out}}$ represents the wave amplitude on the ith edge when leaving the vertex. This vector is transformed into a set of incoming waves $\mathbf{\Psi}_{\text{in}}$ at the other end of the edge by

$$\mathbf{\Psi}_{\text{in}} = \mathbf{D}(k)\mathbf{\Psi}_{\text{out}} \qquad \text{with} \qquad d_{ij} = e^{ikL_i}\delta_{ij}$$

where k represents the wave number. The non-trivial part of the dynamics is contained in the global scattering matrix $\mathbf{S}$ with local scattering amplitudes s_{ij} describing transitions between incoming and outgoing edges i and j, that is,

$$\mathbf{\Psi}_{\text{out}} = \mathbf{S}\mathbf{\Psi}_{\text{in}} = \mathbf{S}\mathbf{D}(k)\mathbf{\Psi}_{\text{out}} = \mathbf{U}(k)\mathbf{\Psi}_{\text{out}} \tag{3}$$

which in turn defines the propagator $\mathbf{U}(k)$ of the discrete wave dynamics on the network. In the following, I will assume that the graph represents a closed system without edges leading to infinity or absorbing vertices. In order to represent true wave propagation, I therefore demand that $\mathbf{S}$ and thus $\mathbf{U}$ are unitary. This implies restrictions on the possible topologies of the graph as the number of incoming and outgoing vertices at each edge must be equal, see Pakoński *et al* (2002). Stationary solutions of the wave dynamics are obtained for wave numbers k fulfilling the secular equation

$$\det(\mathbf{1} - \mathbf{U}(k)) \overset{!}{=} 0. \tag{4}$$

Further restrictions on the dynamical properties on the graph, that is, on the possible transition probabilities $|s_{ij}|^2$, follow from the unitarity condition, see e.g. Pakoński *et al* (2001). I will furthermore assume here that the scattering matrix $\mathbf{S}$ is independent of the wavelength k.

Statistical properties of quantum systems are linked to the complexity of underlying classical dynamics. For quantum systems acting on a continuous coordinate space with continuous or discrete time variable, the classical dynamics revealed in the semiclassical limit is generated by a set of Hamilton's equations of motion or for discrete times by a symplectic map. For a quantum graph, the wave dynamics takes place on discretized time and space variables and a connection to an underlying classical motion is less obvious. The best one can achieve in these circumstances is to identify the quantum propagation on a given graph with a Markov process. The probabilistic 'classical' dynamics is generated by a stochastic transition matrix $\mathbf{T}$ with matrix elements given as (Kottos and Smilansky 1997, Barra and Gaspard 2001, 2002)

$$t_{ij} = |u_{ij}|^2 = |s_{ij}|^2 \tag{5}$$

where t_{ij} denotes the transition probability to jump from an edge i to an edge j at a given vertex. Transition matrices which are derived from a unitary matrix via condition (5) are called unitary stochastic (**Berman and Plemmons 1994**). A unitary stochastic matrix is also doubly stochastic, that is, the rows and columns of $\mathbf{T}$ add up to 1. This implies that $\mathbf{T}$ has the largest eigenvalue equal to 1 with corresponding left and right eigenvectors $\frac{1}{N}(1, 1, 1, \ldots, 1)$. Note, that the converse is not true, that is, not every doubly stochastic matrix is also unitary stochastic.

This leads to the question of how to define the semiclassical limit of a quantum graph. The short wavelength limit $k \to \infty$ for fixed network size N corresponds to a simple scaling transformation, that is, the statistical properties of the spectrum are independent of k.

Instead, a 'semiclassical limit' for quantum graphs must be defined by considering families of graphs of increasing size. Such a limit should be taken so that the stochastic motion converges towards the dynamics of a deterministic map within a given phase space resolution. This semiclassical limit constitutes, therefore, at the same time a classical (deterministic) limit of the stochastic dynamics. Sufficient conditions for constructing such a limit will be formulated at a later stage.

The wave number k defines a one-parameter family of unitary matrices $\mathbf{U}(k)$ for a fixed quantum graph. A stationary solution exists for k-values, for which condition (4) holds, that is, whenever an eigenvalue $\exp(\mathrm{i}\phi_i(k))$ of $\mathbf{U}(k)$ crosses the real line at $+1$. Correlations in the k-spectrum are thus related to correlations in the eigenphases spectrum $\phi_i(k)$ of $\mathbf{U}(k)$ for fixed k. Instead of studying the k-spectrum directly, one can therefore look at the statistical properties of the eigenphases of $\mathbf{U}(k)$ for fixed k and then average over k.

In the generic case, when the sets of lengths L_i are all rationally independent, the one-parameter family $\mathbf{U}(k)$ covers uniformly a subspace of the unitary group $\mathcal{U}(N)$ with the topology of an (at most) N-dimensional torus. That is, the k average can be replaced by an ensemble average over the ensemble $\mathcal{U}_T$ of unitary matrices defined as

$$\mathcal{U}_T = \{\mathbf{U} \in \mathcal{U}(N) | \mathbf{U} = \mathbf{SD} \quad \text{with} \quad d_{ij} = \mathrm{e}^{\mathrm{i}\varphi_i}\delta_{ij} \quad \varphi_i \in [0, 2\pi) \forall i = 1, \ldots, N\}$$

where the scattering matrix $\mathbf{S}$ is kept fixed and the integration measure for the ensemble is given by

$$\mathrm{d}\mu = \prod_{i=1}^{N} \frac{\mathrm{d}\varphi_i}{2\pi}.$$

Note that the ensemble average corresponds to averaging over the spectrum of *a specific quantum system with fixed classical dynamics*. All the unitary matrices forming the ensemble are indeed linked to the same stochastic process on the graph described by a unitary stochastic transition matrix $\mathbf{T}$. I will therefore denote these ensembles as *unitary stochastic ensembles* $\mathcal{U}_T$ (USEs) (Tanner 2001)[1].

It has been argued by Tanner (2001) that correlations in the eigenphases spectrum after ensemble averaging depend crucially on the spectral gap of $\mathbf{T}$, that is, on the distance of the second largest eigenvalue of $\mathbf{T}$ from the unit circle. More precisely, it was conjectured that the spectral correlations follow random matrix theory in the classical limit if the gap decreases slower than $1/N$ for $N \to \infty$. I will study in the following a different statistical measure in more detail, namely the autocorrelation function for spectral determinants.

3. The autocorrelation function for the spectral determinant

The averaged autocorrelation function of the spectral determinant (1) may be defined in terms of the generating function for the square moduli of the secular coefficients, that is,

$$P(z) = \sum_{n=0}^{N} \langle |a_n|^2 \rangle_{\mathcal{U}_T} z^n \tag{6}$$

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[1] It should be noted that a USE is not uniquely defined by a unitary stochastic matrix $\mathbf{T}$ for $N > 2$; there are, in general, several unitary matrices $\mathbf{S}$ not connected by multiplication with diagonal matrices which correspond to the same transition matrix $\mathbf{T}$. The statistical properties of all these ensembles are, however, expected to be identical in the limit $N \to \infty$.

where the average is taken over a unitary stochastic ensemble as defined above. One obtains

$$\tilde{C}(\omega) = e^{-i\frac{N}{2}\omega} \frac{\left\langle \int_0^{2\pi} d\theta\, Z\left(\theta + \frac{\omega}{2}\right) Z^*\left(\theta - \frac{\omega}{2}\right)\right\rangle_{\mathcal{U}_T}}{\left\langle \int_0^{2\pi} d\theta\, |Z(\theta)|^2\right\rangle_{\mathcal{U}_T}} = e^{-i\frac{N}{2}\omega} \frac{P(e^{i\omega})}{P(1)}. \tag{7}$$

It is convenient to rescale the argument so that the correlation function is periodic with period 1, that is, one considers $C(x) = \tilde{C}\left(\frac{2\pi x}{N}\right)$. The symmetry property (2) ensures

$$\langle |a_n|^2\rangle = \langle |a_{N-n}|^2\rangle \qquad \text{and thus} \qquad C(x) = C(1-x). \tag{8}$$

Averages will always be taken over USEs unless stated otherwise and I will drop the suffix $\langle .\rangle_{\mathcal{U}_T}$ from now on. One obtains immediately

$$\langle a_n\rangle = 0 \qquad \forall n = 0, \ldots, N$$

when averaging over the phases φ_i. The variance of the secular coefficients $\langle |a_n|^2\rangle$ has been calculated in the context of RMT by Haake *et al* (1996). In particular one obtains $\mathcal{U}(N)$ (CUE) after averaging over the whole unitary group, over the ensemble of symmetric unitary matrices (COE) or over the ensemble of diagonal unitary matrices (Poisson)

$$\langle |a_n|^2\rangle = \begin{cases} 1 & \text{CUE} \\ 1 + \frac{n(N-n)}{N+1} & \text{COE} \\ \binom{N}{n} & \text{Poisson.} \end{cases} \tag{9}$$

It will be shown in the following that the ensemble average for USEs without symmetries may deviate from the CUE result above. Results for systems with an antiunitary symmetry, as for example time-reversal symmetry, related to COE statistics, will be discussed briefly at the end of section 4.

Using Laplace expansion of the determinant, one writes

$$\det(\mathbf{1} - z\mathbf{U}) = 1 + \sum_{n=1}^{N} z^n \sum_{\alpha \in \Gamma(n,N)} (-1)^n \det(\mathbf{U}[\alpha])$$

where $\Gamma(n, N)$ denotes the set of integer vectors with

$$\Gamma(n, N) = \{\alpha \in \mathbb{N}^n | 1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_n \leqslant N\}$$

and $\mathbf{U}[\alpha]$ is the $n \times n$ matrix obtained from $\mathbf{U}$ by taking only the rows and columns of $\mathbf{U}$ with indices $\alpha_1, \ldots, \alpha_n$, that is,

$$u[\alpha]_{ij} = u_{\alpha_i \alpha_j}.$$

The variance of the distribution of secular coefficients can thus be written as a double sum over all possible subgraphs spanned by the edges $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$\langle |a_n|^2\rangle = \sum_{\alpha,\alpha' \in \Gamma(n,N)} \langle \det(\mathbf{U}[\alpha]) \det(\mathbf{U}^\dagger[\alpha'])\rangle$$

$$= \sum_{\alpha \in \Gamma(n,N)} \langle \det(\mathbf{U}[\alpha]) \det(\mathbf{U}^\dagger[\alpha])\rangle. \tag{10}$$

In the last step, the fact that contributions with $\alpha \neq \alpha'$ vanish identically when taking the ensemble average has been exploited. Using Laplace expansion again one writes each of the sub-determinants in (10) as a sum over permutations π of the row (or column) indices, that is,

$$\det(\mathbf{U}[\alpha]) = \sum_{\pi} (-1)^\pi \prod_{i=1}^{n} u[\alpha]_{i,\pi(i)} = \sum_{\pi} (-1)^\pi u[\alpha]_\pi.$$

The sum over permutations may be interpreted as a sum over closed (periodic) paths on the subgraph α which visit each edge α_i, $i = 1, \ldots, n$ exactly once. These periodic paths will be called covering orbits in what follows; they are not necessarily connected. The product of determinants in (10) now takes on the form

$$\langle |a_n|^2 \rangle = \sum_{\alpha \in \Gamma(n,N)} \left\langle \sum_{\pi,\pi'} (-1)^{\pi+\pi'} u[\alpha]_\pi u[\alpha]_{\pi'}^* \right\rangle.$$

Periodic orbit pairs with $\pi = \pi'$ will obviously survive the ensemble average and may form an important contribution to the double sum. Splitting these diagonal contributions from the off-diagonal contributions and assuming that there are no systematic periodic orbit degeneracies due to symmetries, one obtains

$$\frac{1}{n!} \frac{d^n}{dz^n} P(z)|_{z=0} = \langle |a_n|^2 \rangle = \sum_{\alpha \in \Gamma(n,N)} \sum_\pi t[\alpha]_\pi + \left\langle \sum_{\pi \neq \pi'} (-1)^{\pi+\pi'} u[\alpha]_\pi u[\alpha]_{\pi'}^* \right\rangle$$

$$= \sum_{\alpha \in \Gamma(n,N)} \text{per}(\mathbf{T}[\alpha]) + \left\langle \sum_{\pi \neq \pi'} (-1)^{\pi+\pi'} u[\alpha]_\pi u[\alpha]_{\pi'}^* \right\rangle. \qquad (11)$$

Here, $t[\alpha]_\pi = |u[\alpha]_\pi|^2$ denotes the product of transition probabilities along the periodic path. Furthermore, $\mathbf{T}[\alpha]$ is the sub-matrix obtained from the rows and columns $\alpha_1, \ldots \alpha_n$ of the stochastic transition matrix $\mathbf{T}$ and $\text{per}(\mathbf{T}[\alpha])$ is the permanent of this matrix. Note, that the permanent of a square matrix $\mathbf{A}$ is defined as

$$\text{per}(\mathbf{A}) = \sum_\pi \prod_{i=1}^n a_{i,\pi(i)}.$$

A detailed account of the properties of permanents may be found in Minc (1978), Berman and Plemmons (1994) or Brualdi and Ryser (1991).

One finally obtains for the generating function (6)

$$P(z) = \sum_{n=0}^N \langle |a_n|^2 \rangle z^n = \text{per}(\mathbf{1} + z\mathbf{T}) + \text{non-diagonal contributions}. \qquad (12)$$

The autocorrelation function of the spectral determinant can thus be linked to the permanent of sub-matrices of the stochastic transition matrix $\mathbf{T}$. This is in analogy with the connection between the spectral correlation function and the traces of powers of $\mathbf{T}$ (Tanner 2001). Note that additional contributions to the diagonal approximation have to be taken into account in the presence of symmetries as for example time-reversal symmetry; I will consider this case at the end of the next section.

Two questions arise immediately: firstly, what is the range of validity of the diagonal approximation, and secondly, how can one extract at least asymptotic results in the large N limit from expression (12) by, for example, making a connection to the spectrum of the transition matrix $\mathbf{T}$? The last point especially turns out to be quite tricky; dealing with permanents is a very hard problem in general (unlike calculating determinants), the effort of computing a permanent increases faster than exponential with the matrix size[2]. I will give answers to both the points raised above in the next section.

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[2] Calculating the permanent of a general matrix is an NP complete problem in the language of algorithmic complexity, that is, there is no (known) polynomial algorithm to compute the permanent; see Brualdi and Ryser (1991), as well as Minc (1978) for a description of the best known algorithms.

4. Asymptotic results for chaotic dynamics on graphs

When studying the autocorrelation function in the limit of large network size, one first has to specify, how to take this limit, that is, how to choose families of graphs which may be said to have a well-defined classical and thus semiclassical limit for $N \to \infty$. Taking a very conservative point of view, I will demand that an increase of the size of the network has to be done in a way which leaves the probabilistic dynamics of the Markov process and thus the periodic orbit structure invariant up to a certain cut-off time $n_c(N)$; the cut-off time itself needs to increase with N, one can typically achieve $n_c \sim \log N$. Such a definition ensures automatically that typical measures of chaos such as the topological entropy h_t, the K-entropy or the spectral gap converge for large N. Only graph families without symmetries and with a finite gap and such exponential decay of correlations will be considered in what follows.

The transition matrices of such families of graphs with a chaotic classical limit become increasingly sparse for large N; the number of nonzero matrix elements increases typically not faster than N. This in turn implies that most of the subgraphs α of size n entering the sum (10) contain no or at least one covering orbit for small n. (I will make no distinction between connected and non-connected orbits here.) The diagonal approximation is obviously exact in these cases. Non-diagonal contributions will become important for n values for which two or more covering orbits exist in a typical subgraph. A transition between these two regimes occurs roughly at times which coincide with the radial size of the network, that is, with the mean time to reach each edge from every other edge. This transition time n_t is of the order

$$n_t(N) \sim \frac{\log N}{h_t}$$

where h_t is the topological entropy. The diagonal approximation can thus be estimated to be valid for coefficients $\langle |a_n|^2 \rangle$ with $n < n_t$.

A connection between the permanent appearing in (12) and the spectrum of the transition matrix can be established using similar arguments. Consider first the identity

$$\det(\mathbf{1} - z\mathbf{T}) = \prod_p (1 - z^{n_p} t_p) \tag{13}$$

for determinants, where the product runs over *all* primitive periodic orbits of the graph, that is, connected periodic paths not including repetitions. Here, n_p denotes the length of the orbit and the weight t_p corresponds to the product of transition probabilities along the orbit. To calculate the permanent, one might now be tempted to just replace the minus sign on the right-hand side in (13) by a plus sign. That is, one may consider the function

$$\zeta_+(z) = \prod_p (1 + z^{n_p} t_p) = \sum_{n=0}^{\infty} z^n c_n \tag{14}$$

to obtain a periodic orbit expression for $\text{per}(\mathbf{1} + z\mathbf{T})$. The product in (13) does, however, include self-intersecting periodic paths on the graph, that is, periodic orbits which visit the same edge more than once. It is one of the magical properties of the determinant, that, after expanding the product in (13), contributions of intersecting orbits cancel exactly, leaving only terms containing orbits and products of orbits, which have no edge in common. The same is obviously not true for expression (14). One can argue, however, that short periodic orbits on the graph are in general irreducible, i.e. they do not self-intersect (Bogomolny 1992, Tanner and Wintgen 1992). Furthermore, taking the classical limit in the sense described above corresponds to enhancing the phase space resolution, that is, self-intersecting orbits turn eventually into irreducible orbits when increasing N. Again, self-intersections are likely to occur only for times n larger than the transition time n_t. The coefficients of a polynomial

expansion of the permanent in (12) can thus be approximated by the coefficients of $\zeta_+(z)$ exactly in the regime, where the diagonal approximation holds, that is,

$$\langle |a_n|^2 \rangle \approx \frac{1}{n!} \frac{d^n}{dz^n} \mathrm{per}(\mathbf{1} + z\mathbf{T})|_{z=0} \approx \frac{1}{n!} \frac{d^n}{dz^n} \prod_p (1 + z^{n_p} t_p)|_{z=0}$$

$$\text{for} \qquad n < n_t \sim \frac{\log N}{h_t}. \tag{15}$$

The zeta function (14) can be linked to determinants involving the transition matrix by observing

$$\zeta_+ = \prod_p (1 + z^{n_p} t_p) = \exp\left[\sum_p - \sum_{r=1}^\infty \frac{(-1)^r}{r} z^{rn_p} t_p^r \right]$$

$$= \exp\left[\sum_{r=1}^\infty \frac{1}{r} \left(\sum_p (z^{n_p} t_p)^r - \sum_p (z^{n_p} t_p)^{2r} \right) \right]$$

$$= \frac{\det(\mathbf{1} - z^2 \mathbf{T}^{(1)})}{\det(\mathbf{1} - z\mathbf{T})}. \tag{16}$$

The matrix $\mathbf{T}^{(1)}$ is defined by squaring the matrix elements of $\mathbf{T}$, that is, $t_{ij}^{(1)} = t_{ij}^2$. Equation (16) is the main result of this paper. It differs from the relation given by Kettemann *et al* (1997) by having a determinant also in the numerator; this term arises by including the repetitions of periodic orbits in a consistent manner and must not be neglected. It leads to significant contributions in the expansion of ζ_+ and thus to the coefficients $\langle |a_n|^2 \rangle$ in general. These additional contributions have also been noted by Keating (1996), Cheung (1997) and Snaith (2000). Their results, derived from semiclassical periodic orbit expansions of the spectral determinant, are more general in the sense that they apply for generic quantum systems. The resulting periodic orbit formula equivalent to (16) could, however, not be linked to classical operators. The periodic orbit formulae are notoriously difficult to calculate and specific values for the secular coefficients could indeed not be obtained so far.

Here, one may proceed as follows; after using Newton's method (Kettemann *et al* 1997) to expand the two determinants in (16) in terms of traces, one recovers the coefficients c_n of ζ_+ in equation (14) recursively by applying standard formulae for fractions of power series (Gradshteyn and Ryzhik 1994). The c_n converge exponentially fast to a finite value for large n for $\mathbf{T}$ matrices with a finite spectral gap. The asymptotic behaviour is in particular determined by the residiuum of $\zeta_+(z)$ at the leading singularity at $z = 1$. It follows from relation (15) which is valid over a range of n values increasing with N that the coefficients $\langle |a_\tau|^2 \rangle$ converge to a constant value α for fixed $\tau = n/N$ and $N \to \infty$ given as

$$\alpha = \lim_{N\to\infty} \langle |a_\tau|^2 \rangle = \lim_{N\to\infty} \mathrm{Res}_{z=1} \frac{\det(\mathbf{1} - z^2 \mathbf{T}^{(1)})}{\det(\mathbf{1} - z\mathbf{T})}. \tag{17}$$

This result differs from the CUE result (9) insofar that the constant α is in general not equal to 1. It depends furthermore on the asymptotic limit of the eigenvalue spectra of both $\mathbf{T}$ and $\mathbf{T}^{(1)}$ and thus on the classical limit of the family of graphs considered. Let me demonstrate this point with the help of an example.

Example. Consider the sawtooth map on the unit interval with k legs of slope k. The map has a simple Markov partition whose lth refinement divides the unit interval into k^l subintervals of equal length. The Markov partition defines a transition matrix $\mathbf{T}_k(N)$ of dimension $N = k^l$ with k^{l+1} nonzero matrix elements all equal to $1/k$. The so-defined transition matrices are unitary stochastic and correspond to a graph where each vertex has k incoming and k outgoing edges. The spectra of $\mathbf{T}_k$ and $\mathbf{T}_k^{(1)}$ consist of a single nonzero eigenvalue being 1 and $1/k$,

respectively, independent of the order of refinement l. Transition matrices for fixed k form a family with well-defined classical limit. Following equations (16) and (17) one expects for the coefficients of the autocorrelation function

$$\langle |a_0|^2 \rangle = 1 \qquad \langle |a_1|^2 \rangle = 1 \qquad \alpha = \langle |a_n|^2 \rangle = 1 - \frac{1}{k} \qquad \text{for} \qquad 1 < n < \frac{N}{2}$$

in the limit $N \to \infty$. (The function ζ_+ does of course not obey the symmetry conditions (16), which had to be put in by hand resulting in the cut-off at $n = N/2$.)

In the next section, I will consider a non-trivial example in more detail, namely the so-called binary graphs (Tanner 2000, 2001). Before doing so I will briefly discuss the COE case. Let $U(k)$ defined in (3) have an antiunitary symmetry and assume that a corresponding unitary stochastic matrix ensemble has been chosen which preserves this symmetry. The simplest examples thereof are undirected graphs as considered by Kottos and Smilansky (1997) for which the transitions between incoming and outgoing edges are invariant under reversing the direction on each edge. (Note that here also U is in general not symmetric.) A periodic path on the graph is now either invariant under the antiunitary symmetry (up to cyclic permutations) or is mapped onto a different path with the same weight $u[\alpha]_\pi$. In deriving the formula equivalent to equation (15) in the COE case, one follows essentially the line of arguments laid out in the last section. The main difference is that the double sum in (10) collapses after ensemble average to a single sum including two terms, that is

$$\langle |a_n|^2 \rangle = \sum_{\alpha \in \Gamma(n,N)} \langle \det(\mathbf{U}[\alpha]) \det(\mathbf{U}^\dagger[\alpha]) \rangle + \langle \det(\mathbf{U}[\alpha]) \det(\mathbf{U}^\dagger[A(\alpha)]) \rangle$$

where $A(\alpha)$ is the subgraph related to α by the antiunitary symmetry. When performing the diagonal approximation (11) one needs to take into account that a covering orbit and its symmetry-related partner consisting of two or more disconnected orbits does contribute through all combinations which can be formed between the orbits. These contributions combine in a nice way to replace $\mathrm{per}(\mathbf{1} + z\mathbf{T})$ by $(\mathrm{per}(\mathbf{1} + z\mathbf{T}))^2$ in (12) apart from additional contributions from orbits invariant under the symmetry. One finally obtains for the coefficients $\langle |a_n|^2 \rangle$ in the COE case

$$\langle |a_n|^2 \rangle \approx \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \frac{(\zeta^+(z))^2}{\prod_{sp}(1 + z^{n_{sp}} t_{sp})} = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{\det(\mathbf{1} - z^2 \mathbf{T}^{(1)})}{\det(\mathbf{1} - z\mathbf{T})} \right)^2 \frac{1}{\prod_{sp}(1 + z^{n_{sp}} t_{sp})} \qquad (18)$$

again valid for $n < n_t$. Here, the index sp refers to periodic paths invariant under the symmetry; the product over these orbits has to be included to compensate for over-counting them when taking the square of ζ^+. It is this product which makes equation (18) less accessible than its CUE-counterpart as there is no obvious way to rewrite it in terms of determinants of known matrices.

5. Numerical results

The model systems considered are graphs with an even number of edges and transition matrices of the form

$$t_{ij} = \begin{cases} \frac{1}{2}(\delta_{2i,j} + \delta_{2i+1,j}) & \text{for} \quad 0 \leqslant i < \frac{N}{2} \\ \frac{1}{2}(\delta_{2i-N,j} + \delta_{2i+1-N,j}) & \text{for} \quad \frac{N}{2} \leqslant i < N \end{cases} \qquad i = 0, \ldots, N-1. \qquad (19)$$

The graphs represent for dimensions $N = 2^l$ the Markov partition of a two-leg sawtooth map with complete binary symbolic dynamics. This is a special case of the example above for $k = 2$. I will refer to graphs of the form (19) as binary graphs. The transition matrices (19) are
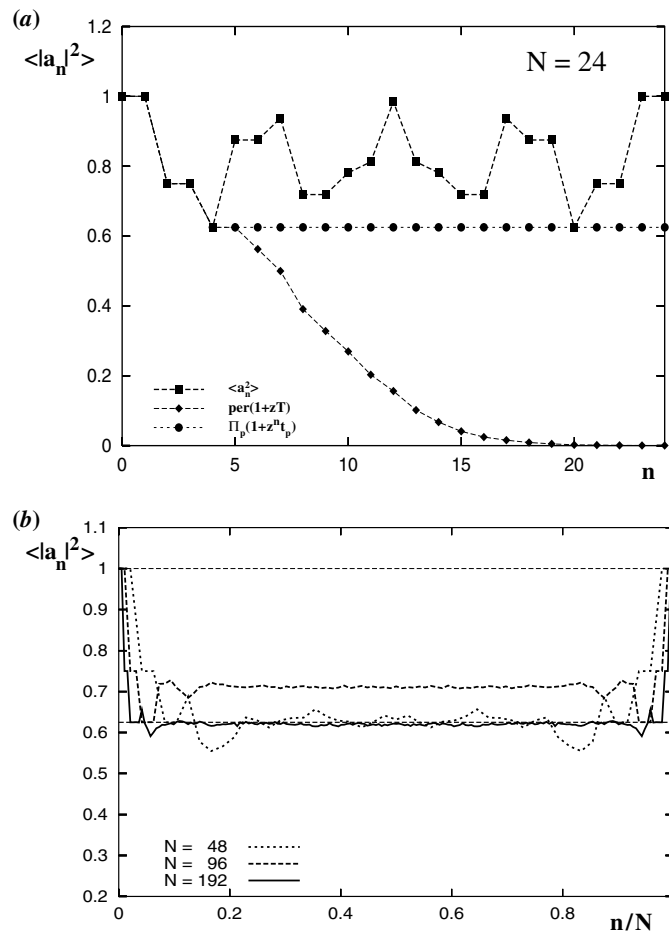
Figure 2. Coefficients of the autocorrelation function for the binary graph family with $p = 3$; (a) coefficients $\langle|a_n|^2\rangle$ compared with the diagonal approximation obtained from the coefficients of $\text{per}(\mathbf{1} + z\mathbf{T})$ and the periodic orbit formula $\zeta_+ = \prod_p (1 + z^{n_p} t_p)$ for $N = 24$; the asymptotic result for this family is 0.625; (b) convergence to the asymptotic result for members of the same family with larger network size; the random matrix result is $\langle|a_n|^2\rangle = 1$.

unitary stochastic; one can show furthermore that binary graphs with dimensions $N = p2^l$ for fixed p, odd, form a family with a well-defined classical limit. The spectral gap is $\Delta = 1/2$ for $p \neq 1$ and $\Delta = \infty$ for $p = 1$. The spectrum of $\mathbf{T}$ (as well as $\mathbf{T}^{(1)}$) depends on p, but not on the order l.

In figure 2(a), the secular coefficients $\langle|a_n|^2\rangle$ obtained from (10) are shown together with the coefficients of $\text{per}(\mathbf{1} + z\mathbf{T})$ and $\zeta_+(z)$ for a member of the $p = 3$ family with dimension $N = 24$. The coefficients of the various approximations all coincide for $n \leqslant 5 \sim n_t = \log 24/\log 2$. The coefficients of the permanent function decrease rapidly thereafter, whereas those of the periodic orbit product approach a constant (which is 0.625 for this particular family). The coefficients of the autocorrelation function still fluctuate wildly for such a small N value. Figure 2(b) shows various members of the same family for larger N values. The fluctuations decrease considerably and convergence to the limiting value $\alpha = 0.625$ is observed. The asymptotic value depends on the family and thus on p; one

observes in particular convergence to the asymptotic value $\alpha = 1/2$ for $p = 1$ as predicted in the example in the previous section.

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